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A QUASI-FLOW CORNER THEORY OF ELASTIC–PLASTIC FINITE DEFORMATION

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Abstract—A quasi-flow corner theory of elastic–plastic finite deformation of ductile materials has been proposed. By introducing a decreasing function of quasi-elastic modulus with respect to strain into the classical flow theory and normality law and by modifying the common decomposition of elastic–plastic strain rate, the present quasi-flow corner theory achieves smooth and continuous transitions from the normality law (the Prandtl–Reuss equation) to the non-normality law with strain, and from plastic loading to elastic unloading. On isotropic condition, the J_2 flow and deformation theories can be included as special cases of the quasi-flow corner theory. The proposed theory is then applied to simulate the instability and deformation localization under plane strain tension and uniaxial tension of anisotropic sheet metals. Some of the numerical results have been compared with experimental ones. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

There has been great interest in the subject of plastic instability and deformation localization for elastic–plastic solids in the last two or three decades. Because the deformation localization observes strong nonlinear characteristics and involves complicate loading and unloading histories, the theoretical analysis and the numerical simulation of the instability and fracture for ductile materials often encounter many difficulties. One of the difficulties is that the instability and localization deformation is sensitive to the constitutive law used in the analysis. To find an efficient constitutive description, the objective co-rotational rate of stress tensor should be first considered. According to the important fact that any possible objective corotative rate is a particular case of the Lie derivative (Marsden and Hughes, 1983), three kinds of typical descriptions of the objective corotative rate for the elastic and the plastic spin referred to deformed configuration were given by Green and Naghdi (1965), Hill and Rice (1972), Duszek and Perzyna (1991, 1993), and Schieck and Stumpf (1995), which are the well-known Zaremba–Jaumann rate, the Green–Naghdi rate and the appropriate co-rotational rate of Schieck–Stumpf. In these corotative rates, the ZJ-rate is invariant with respect to rigid body motion and yields a good approximation for moderately large strains, the SS-rate seems to valid for the whole range of infinite elastoplasticity, while the GN-rate is identical with the SS-rate for the special cases of hypoelasticity and rigid-plasticity.

Another difficulty is how to correctly find the constitutive models involving the evolution equation of plastic deformation history. In recent years, several constitutive theories and models based on the ZJ-rate have been proposed in order to overcome the special difficulties in this subject.

Budiansky (1959), Stören and Rice (1975), and Hutchinson (1974) developed a vertex hardening constitutive model based on the incremental form of the J_2 deformation theory, which provides more accurate predictions for the buckling process of sheet metal than the classical flow theory of plasticity does. Christofferson and Hutchinson (1979) proposed a class of plastic constitutive equations with vertex effect in which the plastic potential is introduced as a function of stress increments, and a smooth transition from plastic loading to elastic unloading is incorporated. Its simplest form called J_2 corner theory was then applied to several localization problems. Recently, Gotoh (1985) proposed a group of plastic constitutive equations in the form of tensor algebra, which show that the vertex effect satisfies mathematical restriction about tensor functions. A common feature of the above theories is that the non-orthogonality rule between plastic strain increment and yield surface is observed throughout the whole plastic deformation process from initial yielding up to final localization and fracture.

From the viewpoint of plastic potential and classical flow theory, it is generally acknowledged that in the initial stage of plastic deformation, the yield surface should be convex and smooth and consistent with the normality rule of plastic flow. However, instability and localization of plastic deformation always occur at the final stage of plastic deformation where the succeeding "yield surface" will gradually appear as a vertex effect. For this stage, a non-normality rule is needed to describe the plastic deformation and continuous yield surface.

The purpose of the present paper is to modify the classical flow theory in order to include the non-normality rule, while plastic deformation approaches the stage of instability and localization. Therefore, a relatively simple phenomenological quasi-flow corner theory for elastic–plastic finite deformation will be proposed. By introducing an instantaneous quasi-elastic modulus and a modulus evolutional function with strain, a continuous and smooth transition from the normality rule to the non-normality rule with vertex effect and from the plastic loading to the elastic unloading after the plastic instability will be realized. The quasi-flow constitutive description can make any anisotropic yielding rule without vertex hardening combined with vertex hardening, and connect the classical J_2 flow theory with J_2 deformation theory on isotropic hardening condition. It should be mentioned that in order to compare the result of the present theory with that of the above consitutive theories, the ZJ-rate is also used in the constitutive formulation.

In Section 2.1, to simplify formulae deduction, the small strain version of the quasi-flow corner theory described is used for reference of the formulation scheme of the corner theory proposed by Christofferson and Hutchinson (1979). In Section 2.2, the quasi-elastic modulus E_Q in the proposed theory is determined and described in detail, which is the main character of the present quasi-flow corner theory. In Section 2.3, we discuss the differences of the constitutive theories by comparing the present theory with other corner theories based on the isotropic Mises yield criterion. Section 2.4, extends the small strain version into finite strain problems. Numerical simulation based on the proposed theory and comparison within the theory and other corner theories as well as some available experimental results are then presented in Section 3.

2. BASIC EQUATIONS FOR QUASI-FLOW CORNER THEORY

2.1. *Small deformation formulation*

For general strain hardening materials, a convex yield surface is assumed The total strain rate component $\dot{\varepsilon}_{ij}$ can be divided into two parts:

$$\dot{\varepsilon}_{ij} = \dot{\bar{\varepsilon}}^{e}_{ij} + \dot{\bar{\varepsilon}}^{p}_{ij} \tag{1}$$

where $\dot{\bar{\varepsilon}}^{e}_{ij}$ is defined as quasi-elastic strain rate and $\dot{\bar{\varepsilon}}^{p}_{ij}$ as quasi-plastic strain rate. For an arbitrary yield function f, $\dot{\bar{\varepsilon}}^{p}_{ij}$ is assumed to obey a quasi-flow rule:

$$\dot{\bar{\varepsilon}}^{p}_{ij} = \dot{\lambda}_{Q} \frac{\partial f}{\partial \sigma_{ij}} \tag{2}$$

where σ_{ij} is the Cauchy stress and $\dot{\lambda}_Q$ is a scalar factor, which may be positive on plastic loading condition or equal to zero on elastic unloading condition. Furthermore, we define the elastic modulus in instantaneous elastic compliance tensor as:

$$E_Q(\bar{\varepsilon}^p) = E * g(\bar{\varepsilon}^p) \tag{3}$$

where $g(\bar{\varepsilon}^p)$ is a modulus evolutional function with respect to the current plastic effective strain $\bar{\varepsilon}^p$, and E is the Young modulus. Under the condition of constant volume law of plastic deformation, a quasi Poisson ratio (μ_Q) is introduced:

$$\frac{1-2\mu_Q}{E_Q} = \frac{1-2\mu}{E} = \text{constant} \tag{4}$$

where μ is the common Poisson ratio. From the above equations, one obtains:

$$\mu_Q = \tfrac{1}{2} - g(\bar{\varepsilon}^p)(\tfrac{1}{2} - \mu). \tag{5}$$

Introducing eqns (3) and (5) into the instantaneous elastic compliances $\tilde{D}^e_{ijkl}$, we obtain a series of relations as follows:

$$\dot{\sigma}_{ij} = \tilde{D}^e_{ijkl}\dot{\varepsilon}^e_{kl} \tag{6}$$

$$\dot{\sigma}^p_{ij} = \tilde{D}^e_{ijkl}\dot{\varepsilon}^p_{kl} \tag{7}$$

$$\dot{\sigma}^e_{ij} = \tilde{D}^e_{ijkl}\dot{\varepsilon}_{kl} \tag{8}$$

$$\dot{\sigma}_{ij} = \dot{\sigma}^e_{ij} - \dot{\sigma}^p_{ij} = \tilde{D}^e_{ijkl}\dot{\varepsilon}_{kl} - \dot{\lambda}_Q \tilde{D}^e_{ijkl}\frac{\partial f}{\partial \sigma_{kl}} \tag{9}$$

where $\dot{\sigma}_{ij}$ is the Cauchy stress rate. $\dot{\sigma}^e_{ij}$ and $\dot{\sigma}^p_{ij}$ are the quasi-elastic and quasi-plastic stress rates corresponding to $\dot{\varepsilon}^e_{ij}$ and $\dot{\varepsilon}^p_{ij}$, respectively.

The general yield criterion can be written as:

$$f(\sigma_{ij}, \varepsilon^p_{ij}, \tilde{\kappa}) = 0 \tag{10}$$

where ε^p_{ij} is plastic strain tensor and $\tilde{\kappa}$ is an internal variable which can be related to the assumed quasi-plastic work $\tilde{\omega}_p$ ($\tilde{\omega}_p = \int \sigma_{ij}\, \dot{\varepsilon}^p_{ij}$). According to the consistent condition, one can obtain from eqns (2), (6) and (9):

$$\dot{\lambda}_Q = \frac{\alpha}{A}\left(\frac{\partial f}{\partial \sigma_{ij}}\right)\tilde{D}^e_{ijkl}\dot{\varepsilon}_{kl} \tag{11}$$

where

$$\alpha = \begin{cases} 1 & \text{for plastic loading} \\ 0 & \text{for elastic unloading} \end{cases} \tag{12}$$

$$A = \frac{\partial f}{\partial \sigma_{ij}}\tilde{D}^e_{ijkl}\frac{\partial f}{\partial \sigma_{kl}} - \frac{\partial f}{\partial \varepsilon^p_{ij}}\frac{\partial f}{\partial \sigma_{ij}} - \frac{\partial f}{\partial \tilde{\kappa}}\frac{\partial \tilde{\kappa}}{\partial \bar{\varepsilon}^p_{ij}}\frac{\partial f}{\partial \sigma_{ij}}. \tag{13}$$

Substituting eqn (11) into eqn (2) with considering eqn (1), one obtains the constitutive equation for the quasi-flow theory:

$$\dot{\sigma}_{ij} = (\tilde{D}^e_{ijkl} - \tilde{D}^p_{ijkl})\dot{\varepsilon}_{kl} \tag{14}$$

where

$$\tilde{D}^{p}_{ijkl} = \frac{\alpha}{A}\tilde{D}^{e}_{ijmn}\frac{\partial f}{\partial \sigma_{mn}}\frac{\partial f}{\partial \sigma_{pq}}\tilde{D}^{e}_{pqkl}. \tag{15}$$

On the other hand, we define H_Q as the slope of the stress vs quasi-plastic strain curve under uniaxial tension, i.e.:

$$H_Q = \frac{\dot{\sigma}}{\dot{\varepsilon}^{p}} \tag{16}$$

or

$$\dot{\sigma} = H_Q\dot{\varepsilon}^{p} = E_Q\dot{\varepsilon}^{e}. \tag{17}$$

Because of

$$\dot{\sigma} = E_t\dot{\varepsilon} \tag{18}$$

where E_t is the tangent slope of the stress–strain curve under uniaxial tension, one has:

$$H_Q = \frac{E_Q E_t}{E_Q - E_t}. \tag{19}$$

On the general stress states, we assume that there is the effective quasi-plastic strain rate $\dot{\bar{\varepsilon}}^{p}$ connected with the effective stress rate $\dot{\bar{\sigma}}$ in terms of H_Q, that is:

$$\dot{\bar{\sigma}} = H_Q\dot{\bar{\varepsilon}}^{p}. \tag{20}$$

The quasi-plastic deformation work rate is

$$\dot{\omega}_p = \sigma_{ij}\dot{\varepsilon}^{p}_{ij} = \dot{\lambda}_Q\sigma_{ij}\frac{\partial f}{\partial \sigma_{ij}} \tag{21}$$

or

$$\dot{\omega}_p = \sigma_{ij}\dot{\varepsilon}^{p}_{ij} = \bar{\sigma}\dot{\bar{\varepsilon}}^{p}. \tag{22}$$

For an M-order yield function (f), in consideration of the Euler theorem and the associate flow law, one can obtain:

$$\dot{\lambda}_Q = \frac{\dot{\omega}_p}{Mf} = \frac{\bar{\sigma}}{Mf}\left(\frac{1}{E_t} - \frac{1}{E_Q}\right)\frac{\partial \bar{\sigma}}{\partial \sigma_{ij}}\dot{\sigma}_{ij}. \tag{23}$$

Using the above equations (1), (2), (12) and (23), one obtains:

$$\begin{aligned}\dot{\varepsilon}_{ij} &= \left\{\frac{1}{E_Q}\left[\frac{1}{2}(1+\mu_Q)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) - \mu_Q\delta_{ij}\delta_{kl}\right]\right.\\ &\quad \left. + \frac{\alpha\bar{\sigma}}{Mf}\left(\frac{1}{E_t} - \frac{1}{E_Q}\right)\frac{\partial f}{\partial \sigma_{ij}}\frac{\partial \bar{\sigma}}{\partial \sigma_{kl}}\right\}\dot{\sigma}_{kl}\\ &= (\tilde{C}^{e}_{ijkl} + \tilde{C}^{p}_{ijkl})\dot{\sigma}_{kl}\\ &= \tilde{C}^{ep}_{ijkl}\dot{\sigma}_{kl}.\end{aligned} \tag{24}$$

Because of the unique corresponding relation between $\dot{\sigma}_{ij}$ and $\dot{\varepsilon}_{ij}$, it is sure that eqn (24) is the inversion of eqn (14). Finally, we call eqns (14) and (24) the constitutive equations of the quasi-flow theory.

From eqns (4) and (5), the following relation can be easily obtained

$$\frac{1+\mu_Q}{E_Q} \equiv \frac{1+\mu}{E} + \frac{3}{2}\left(\frac{1}{E_Q} - \frac{1}{E}\right). \tag{25}$$

So, eqn (24) can also be re-expressed as:

$$\begin{aligned}\dot{\varepsilon}_{ij} &= \frac{1}{E}\left[\frac{1}{2}(1-\mu)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) - \mu\delta_{ij}\delta_{kl}\right]\dot{\sigma}_{kl} \\ &\quad + \left[\frac{3}{2}\left(\frac{1}{E_Q} - \frac{1}{E}\right)I_{ijkl} + \alpha\left(\frac{\bar{\sigma}}{Mf}\right)^2\left(\frac{1}{E_t} - \frac{1}{E_Q}\right)\frac{\partial f}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{kl}}\right]\dot{\sigma}_{kl} \\ &= B^{e}_{ijkl}\dot{\sigma}_{kl} + B^{p}_{ijkl}\dot{\sigma}_{kl}\end{aligned} \tag{26}$$

where

$$I_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) - \tfrac{1}{3}\delta_{ij}\delta_{kl} \tag{27}$$

and B^{e}_{ijkl} is the true elastic compliance tensor. So, the true plastic strain rate $\dot{\varepsilon}^{p}_{ij}$ can be determined as:

$$\dot{\varepsilon}^{p}_{ij} = B^{p}_{ijkl}\dot{\sigma}_{kl}. \tag{28}$$

2.2. *Determination of the quasi-elastic modulus* E_Q

It is interesting to note that for Mises yield criterion and by selecting the quasi-elastic modulus $E_Q \equiv E_S$ (E_S is the secant modulus of the uniaxial stress–strain curve at $\bar{\sigma}$), eqn (26) comes back to the rate form of the J_2D theory proposed by Stören and Rice (1975). It should be mentioned that the J_2D theory focuses attention on the vicinity of the critical point of bifurcation and post-bifurcation responses. It is obviously seen from eqn (26) that because $E_S \ll E$ (E is true elastic modulus), the J_2D theory does not obey the normality law of plastic flow during the whole plastic deformation process from initial yielding to fracture. Therefore, J_2D theory is not coincident with the physical fact that before plastic instability, the yield surface is smooth and the direction of the plastic strain rate $\dot{\varepsilon}^{p}_{ij}$ is orthogonal to the loading surface. So, for non-proportional loading cases before instability, the J_2D theory will obtain obviously different results from the classical flow theory (see Section 3.1). Besides, the J_2D theory cannot automatically keep the continuity from plastic loading to elastic unloading because $\dot{\varepsilon}^{p}_{ij}$ in eqn (26) is not equal to zero when elastic unloading starts. The later shortcoming of the J_2D theory has been efficiently overcome by the "J_2-corner theory" (Christofferson and Hutchinson, 1979). However, the J_2-corner theory seems to be not well used to the homogeneous plastic deformation stage where the classical flow theory (J_2F) applies. In consideration of the above problems, the proposed quasi-flow corner theory is designed to realize the smooth and continuous transition from the orthogonality rule to the non-orthogonality rule during plastic loading to elastic unloading. This transition depends on the reasonable determinations of the quasi elastic modulus E_Q and the evolutional functions $g(\bar{\varepsilon}^{p})$.

Based on the above analyses an original scheme to determine the quasi elastic modulis E_Q is proposed. First, we consider the plasticity in the vicinity of bifurcation. Let us also introduce the angle θ defined by the following formula:

$$\cos\theta = \frac{tr(\sigma'_{ij}\dot{\sigma}'_{ij})}{[tr(\sigma'_{ij}\sigma'_{ij})tr(\dot{\sigma}'_{ij}\dot{\sigma}'_{ij})]^{1/2}} \tag{29}$$

where, σ'_{ij} and $\dot{\sigma}'_{ij}$ are the deviator tensors of σ_{ij} and $\dot{\sigma}_{ij}$, respectively. The physical meaning of θ is explained as an angle between σ'_{ij} and $\dot{\sigma}'_{ij}$ in the five-dimensional deviator stress space (Gotoh 1985). Furthermore, we presume that there is an angle θ_0 which divides the plastic loading ranges into two parts: one is the total loading range in which the loading path satisfies the condition of $\theta \leqslant \theta_0$. Now, $\bar{\varepsilon}_1^{p*}$ (θ_0) is defined as the effective strain at the moment of $\theta = \theta_0$, which can be calculated by

$$\bar{\varepsilon}_1^{p*} = \frac{\sigma_{ij}^{*}\varepsilon_{ij}^{p*}}{\bar{\sigma}_1^{*}} \tag{30}$$

where, $\bar{\sigma}_1^*$ is the effective stress corresponding to $\bar{\varepsilon}_1^{p*}$.

In the range of $\bar{\varepsilon}^p \leqslant \bar{\varepsilon}_1^{p*}$, we assume that E_Q can be expressed as:

$$E_Q = E * g_1(\bar{\varepsilon}^p) \tag{31}$$

and design the evolutional function $g_1(\bar{\varepsilon}^p)$ as

$$g_1(\bar{\varepsilon}^p) = \frac{1}{2}\left(1+\frac{E_S}{E}\right) - \frac{1}{\pi}\left(1-\frac{E_S}{E}\right)\tan^{-1}\left(\frac{\bar{\varepsilon}^p - \bar{\varepsilon}_0^{p*}}{a_0}\right) \tag{32}$$

where a_0 is a constant related to the strain hardening exponent n, which can, in general, be taken as $n/20$; $\bar{\varepsilon}_0^{p*}$ is a transitional threshold strain value where E_Q transforms to E_S, and is related to material characters and plastic deformation manners. A simple selecting scheme is to express the $\bar{\varepsilon}_0^{p*}$ using a modified factor C_0, i.e.

$$\bar{\varepsilon}_0^{p*} = C_0 \varepsilon_n^{p*} \tag{33}$$

where C_0 is a constant smaller than unity (taken in the range of $0.7 \sim 0.95$). The meaning of eqn (33) can be explained by the following consideration. Before the material instability of whole macrostructure, the plastic strain rate $\dot{\varepsilon}_{ij}^p$ in some local regions of the macrostructure have deviated normality rule. The micro-description of the deviation is obviously difficult. However, in phenomenological sense, the deviation implies an earlier modulus softening. So, if ε_n^{p*} is thought to be the total average plastic instability strain, $\bar{\varepsilon}_0^{p*}$ can be qualitatively considered as the critical instability plastic strain of internal material points. For example, for the uniaxial tension, the ε_n^{p*} value can be obtained from the solution of the following equation given by Hutchinson and Neale (1977)

$$\frac{\varepsilon_n^{p*}}{n}\exp\left[-\left(\frac{\varepsilon_n^{p*}}{n}-1\right)\right] = (1-f_0)^{1/n} \tag{34}$$

where f_0 is an initial geometrical inhomogeneous parameter.

It can be seen from eqns (26), (31) and (32) that in the initial stage of plastic deformation ($\bar{\varepsilon}^p \ll \bar{\varepsilon}_0^{p*}$), $E_Q = E$. This fact means that eqn (26) returns to the constitutive equation of the classical flow theory coinciding with normality law. As plastic deformation develops, especially after $\bar{\varepsilon} > \bar{\varepsilon}_0^{p*}$, E_Q transforms to E_S, and the vertex effect described by the second term on the right-hand side in eqn (26) will gradually increase. For isotropic Mises materials, eqn (26) comes back to the constitutive equation of the rate form of J_2D theory. The selection of the threshold value ε_n^{p*} should refer to the critical instability plastic strain of materials.

Now, we illustrate the scheme of determining E_Q in the transitional range from plastic loading to elastic unloading. Firstly, it should be noted that an equivalent meaning of the so-called elastic unloading is $\dot{\varepsilon}^p_{ij}$ or $\dot{\bar{\varepsilon}}^p = 0$. Based on such a fact, we assume that there is a conical angle θ_Q of loading surface, whose conical surface separates elastic unloading and plastic flow. For convenience of numerical calculation, we introduce a small constant δ_Q and assume that when $\dot{\bar{\varepsilon}}^p\Delta t \leqslant \delta_Q$, strain rate $\dot{\varepsilon}_{ij}$ falls within the elastic unloading range. The θ value calculated at the moment of $\dot{\bar{\varepsilon}}^p\Delta t = \delta_Q$ can be defined as θ_Q. If $\bar{\varepsilon}^{p*}_1(\theta_0)$ is considered as the effective plastic strain consistent to $\theta = \theta_0$, the quasi-elastic modulus E_Q in the range of $\theta_0 < \theta \leqslant \theta_Q$ is designed as follows

$$E_Q = E * g_2(\bar{\varepsilon}^p) \tag{35}$$

and

$$g_2(\bar{\varepsilon}^p) = \frac{E_S}{E} + \frac{4}{\pi}\left(1 - \frac{E_S}{E}\right)\tan^{-1}\left(\frac{\dot{\bar{\varepsilon}}^p\Delta t}{\delta_Q}\right). \tag{36}$$

From eqn (31) together with (32), we can see that a continuous transition of E_Q from E to E_S is achieved out during loading process; and furthermore from eqn (35) together with (36), we can also find that when $\dot{\bar{\varepsilon}}^p\Delta t = \delta_Q$, i.e. $\bar{\varepsilon}^p(t+\Delta t) = \bar{\varepsilon}^p(t) + \delta_Q$, the expression of eqn (35) will automatically give $E_Q = E$ and $\mu_Q = \mu$ during the unloading process. When the angle θ, calculated by eqn (29), is equal to θ_0, we begin calculating $\dot{\bar{\varepsilon}}^p$ using the following expression:

$$\dot{\bar{\varepsilon}}^p = \dot{\bar{\varepsilon}} - \dot{\bar{\varepsilon}}^e = \frac{\sigma_{ij}(\dot{\varepsilon}_{ij} - \dot{\varepsilon}^e_{ij})}{\bar{\sigma}} \tag{37}$$

where

$$\dot{\varepsilon}^e_{ij} = \frac{1}{E}\left[\frac{1}{2}(1+\mu)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \mu\delta_{ij}\delta_{kl}\right]\dot{\sigma}_{kl} = B^e_{ijkl}\dot{\sigma}_{kl} \tag{38}$$

and $\dot{\sigma}_{ij}$ is calculated by eqn (14).

It must be mentioned that the continuous evolution of the quasi-elastic modulus E_Q from initial plastic deformation up to elastic unloading proves such a fact that the second term of the right-hand side of eqn (26) gradually increases its vertex effect on yield surface at loading point within the total plastic loading range ($\theta \leqslant \theta_0$) as plastic deformation develops; in the transitional range ($\theta_0 < \theta \leqslant \theta_Q$), the influence of the term gradually decreases, and does not eliminate until $E_Q = E$. Besides, recalling eqn (12), eqn (26) will automatically reduce $\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij}$ (i.e. $\dot{\varepsilon}^p_{ij} = 0$), which shows a continuous transition of elastic–plastic strain rate from plastic loading to elastic unloading. The evolutional process of E_Q is depicted in Fig. 1. Taking a comprehensive view of the above discussion, it can be found that by the determination of the rational evolution of the quasi-elastic modulus, E_Q, the

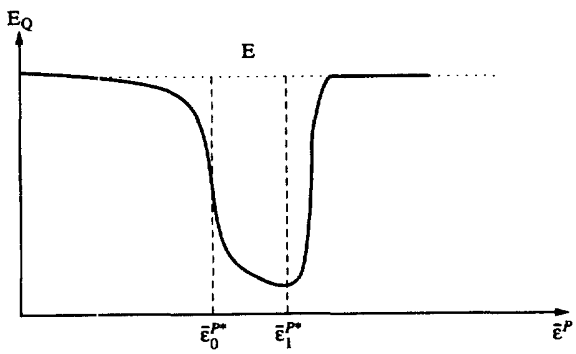


Fig. 1. Evolutional curve of the quasi-elastic modulus E_Q with $\bar{\varepsilon}^p$.

quasi-flow corner theory achieves continuous and smooth transitions from the normality rule (the Prandtl–Reuss equation) to the non-orthogonality rule with strain, and from plastic loading to elastic unloading. In addition, the theory can be extended to the case where generally initial and consequent anisotropy plays an important role, and then, it becomes possible that any anisotropic rule of yielding such as kinematic hardening, kinematic-isotropic hardening and other general anisotropic hardening without vertex effect is combined with vertex hardening. For isotropic Mises hardening materials, *QFC* theory comes back to J_2F theory, when $E_Q \equiv E$, and back to J_2D theory when $E_Q \equiv E_S$.

2.3. *Comparisons within different plastic constitutive theories*

In this section, we focus attention on some J_2 class of corner theories based on the isotropic Mises yield criterion in order to understand the essential character of these constitutive theories. For convenience of comparison, we call the quasi-flow corner theory for isotropic Mises materials QJ_2FC theory.

2.3.1. J$_2$F *theory*. The strain rate separation for J_2F theory is

$$\dot{\varepsilon}_{ij} = B^{e}_{ijkl}\dot{\sigma}_{kl} + \langle\dot{\lambda}_F\rangle\sigma'_{ij} = \left(\frac{1}{2G}I_{ijkl} + \frac{1-2\mu}{3E}\delta_{ij}\delta_{kl}\right)\dot{\sigma}_{kl} + \langle\dot{\lambda}_F\rangle\sigma'_{ij} \tag{39}$$

where

$$\langle\dot{\lambda}_F\rangle = \begin{cases} \dfrac{2\dot{\bar{\sigma}}}{3\bar{\sigma}}\left(\dfrac{1}{E_t} - \dfrac{1}{E}\right), & \text{if plastic loading} \\ 0, & \text{if elastic unloading} \end{cases} \tag{40}$$

in which G and $\dot{\bar{\sigma}}$ are shear modulus and effective Cauchy stress rate, respectively. Noting that

$$J_2 = \tfrac{1}{3}\bar{\sigma}^2 = \tfrac{1}{2}\sigma'_{ij}\sigma'_{ij} \tag{41}$$

and

$$\dot{J}_2 = \sigma'_{ij}\dot{\sigma}'_{ij} \tag{42}$$

we can obtain

$$\dot{\bar{\sigma}} = \frac{3}{2\bar{\sigma}}\sigma'_{ij}\dot{\sigma}'_{ij}. \tag{43}$$

Therefore, we have

$$\dot{\varepsilon}_{ij} = B^{e}_{ijkl}\dot{\sigma}_{kl} + \left(\frac{3}{2\bar{\sigma}}\right)^2\left(\frac{1}{E_t} - \frac{1}{E}\right)\sigma'_{ij}\sigma'_{kl}\dot{\sigma}_{kl}. \tag{44}$$

Using volume strain rule

$$\dot{\varepsilon}_{ii} = \frac{1-2\mu}{E}\dot{\sigma}_{ii} \tag{45}$$

and multiplying σ'_{ij} on the two ends of the above expression, we get

$$\dot{J}_2 = \frac{\sigma'_{ij}\dot{\varepsilon}_{ij}}{(1+\mu)/E+2J_2/h_F} \tag{46}$$

where

$$\frac{1}{h_F} = \left(\frac{3}{2\bar{\sigma}}\right)^2\left(\frac{1}{E_t}-\frac{1}{E}\right). \tag{47}$$

Substituting eqns (45) and (46) into eqn (39), the inverted relation of eqn (39) can be expressed as

$$\dot{\sigma}_{ij} = \frac{E}{1+\mu}\left\{\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\frac{\mu}{1-2\mu}\delta_{ij}\delta_{kl}-\frac{1}{q_F}\sigma'_{ij}\sigma'_{kl}\right\}\dot{\varepsilon}_{kl} \tag{48}$$

in which

$$q_F = \frac{1+\mu}{E}h_F+2J_2. \tag{49}$$

2.3.2. J_2D *theory [proposed by Stören and Rice (1975)].*

$$\dot{\varepsilon}_{ij} = B^e_{ijkl}\dot{\sigma}_{kl}+\frac{3}{2}\left(\frac{1}{E_S}-\frac{1}{E}\right)I_{ijkl}\dot{\sigma}_{kl}+\left(\frac{3}{2\bar{\sigma}}\right)^2\left(\frac{1}{E_t}-\frac{1}{E_S}\right)\dot{J}_2\sigma'_{ij}. \tag{50}$$

Assuming that

$$\frac{1}{2G_S} = \frac{1+\mu_S}{E_S} = \frac{1+\mu}{E}+\frac{3}{2}\left(\frac{1}{E_S}-\frac{1}{E}\right) \tag{51}$$

and

$$\frac{1}{h_S} = \left(\frac{3}{2\bar{\sigma}}\right)^2\left(\frac{1}{E_t}-\frac{1}{E_S}\right) \tag{52}$$

the inversion of eqn (50) is

$$\dot{\sigma}_{ij} = \frac{E_S}{1+\mu_S}\left\{\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\frac{\mu_S}{1-2\mu_S}\delta_{ij}\delta_{kl}-\frac{1}{q_S}\sigma'_{ij}\sigma'_{kl}\right\}\dot{\varepsilon}_{kl} \tag{53}$$

in which

$$q_S = \frac{1+\mu_S}{E_S}h_S+2J_2. \tag{54}$$

2.3.3. J_2C *theory [proposed by Christofferson and Hutchinson (1979)].*

$$\dot{\varepsilon}_{ij} = B^e_{ijkl}\dot{\sigma}_{kl}+\frac{3}{2}\beta\left(\frac{1}{E_S}-\frac{1}{E}\right)I_{ijkl}\dot{\sigma}_{kl}+\left[\beta\left(\frac{1}{E_t}-\frac{1}{E_S}\right)+\gamma\left(\frac{1}{E_t}-\frac{1}{E}\right)\right]\left(\frac{3}{2\bar{\sigma}}\right)^2\dot{J}_2\sigma'_{ij} \tag{55}$$

in which

$$\begin{cases} \beta = f(\theta)[1+\frac{1}{2}f'(\theta)\cot\theta/f(\theta)] \\ \gamma = -\frac{1}{2}f'(\theta)(\sin\theta\cos\theta)^{-1} \end{cases} \tag{56}$$

$$f(\theta) = \begin{cases} 1, & 0 \leqslant \theta \leqslant \theta_0 \\ \cos^2\left[\dfrac{\pi}{2}\left(\dfrac{\theta-\theta_0}{\theta_c-\theta_0}\right)\right], & \theta_0 \leqslant \theta \leqslant \theta_c. \\ 0, & \theta_c < \theta \leqslant \pi \end{cases} \tag{57}$$

$$\cos\theta = \frac{\left(\dfrac{1}{E_t}-\dfrac{1}{E}\right)^{1/2}\dot{\bar{\sigma}}}{\left[\dfrac{3}{2}\left(\dfrac{1}{E_S}-\dfrac{1}{E}\right)\sigma'_{ij}\sigma'_{ij}+\left(\dfrac{1}{E_t}-\dfrac{1}{E_S}\right)\dot{\bar{\sigma}}^2\right]^{1/2}}. \tag{58}$$

Assuming that

$$\frac{1}{h_c} = \left[\beta\left(\frac{1}{E_t}-\frac{1}{E_S}\right)+\gamma\left(\frac{1}{E_t}-\frac{1}{E}\right)\right]\left(\frac{3}{2\bar{\sigma}}\right)^2 \tag{59}$$

$$\frac{1}{2G_c} = \frac{1+\mu_c}{E_c} = \frac{1+\mu}{E}+\frac{3}{2}\beta\left(\frac{1}{E_S}-\frac{1}{E}\right) \tag{60}$$

the inversion of eqn (55) can be obtained using the analogous reduction of J_2F theory

$$\dot{\sigma}_{ij} = \frac{E_c}{1+\mu_c}\left\{\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\frac{\mu_c}{1-2\mu_c}\delta_{ij}\delta_{kl}-\frac{1}{q_c}\sigma'_{ij}\sigma'_{kl}\right\}\dot{\varepsilon}_{kl} \tag{61}$$

in which

$$\begin{cases} E_c = \dfrac{3E}{1-2\mu+E/G_c} \\ \mu_c = \dfrac{E/G_c-2(1-2\mu)}{2[1-2\mu+E/G_c]}. \\ q_c = \dfrac{1+\mu_c}{E_c}h_c+2J_2 \end{cases} \tag{62}$$

2.3.4. J_2G *theory [proposed by Gotoh (1985)].*

$$\dot{\varepsilon}_{ij} = B^e_{ijkl}\dot{\sigma}_{kl}+\frac{a\langle P(\theta)\rangle}{2h_0}I_{ijkl}\dot{\sigma}_{kl}+\frac{b\langle P(\theta)\rangle}{3h_0\cos\theta}\left(\frac{3}{2\bar{\sigma}}\right)^2\dot{J}_2\sigma'_{ij} \tag{63}$$

in which the definition of the angle θ is the same as that in eqn (29), and

$$\begin{cases} h_0 = \dfrac{1}{3}\dfrac{EE_t}{E-E_t} \\ \langle P(\theta)\rangle = a+b\cos\theta \\ a = 1-b = \cos\theta_0/(1+\cos\theta_0) \\ \theta_0 = \dfrac{\pi}{2}-\rho\bar{\varepsilon}^{p^2} \\ \bar{\varepsilon}^p = \int\dot{\bar{\varepsilon}}^p\,\mathrm{d}t, \quad \dot{\bar{\varepsilon}}^{p^2} = \dfrac{2}{3}\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij} \end{cases} \tag{64}$$

in which ρ is a pre-given small constant. Assuming that

$$\frac{1}{2G_g} = \frac{1+\mu_g}{E_g} = \frac{1+\mu}{E}+\frac{a\langle P(\theta)\rangle}{2h_0} = \frac{1+\mu}{E}+\frac{3a}{2}\left(\frac{1}{E_t}-\frac{1}{E}\right) \tag{65}$$

$$\frac{1}{h_g} = \frac{b\langle P(\theta)\rangle}{\cos\theta}\left(\frac{3}{2\bar{\sigma}}\right)^2\left(\frac{1}{E_t}-\frac{1}{E}\right) \tag{66}$$

the inversion of eqn (63) is

$$\dot{\sigma}_{ij} = \frac{E_g}{1+\mu_g}\left\{\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\frac{\mu_g}{1-2\mu_g}\delta_{ij}\delta_{kl}-\frac{1}{q_g}\sigma'_{ij}\sigma'_{kl}\right\}\dot{\varepsilon}_{kl} \tag{67}$$

in which

$$\begin{cases} E_g = \dfrac{3E}{1-2\mu+E/G_g} \\ \mu_g = \dfrac{E/G_g-2(1-2\mu)}{2[1-2\mu+E/G_g]} \\ q_g = \dfrac{1+\mu_g}{E_g}h_g+2J_2. \end{cases} \tag{68}$$

2.3.5. $\mathrm{QJ_2FC}$ *theory*.

$$\dot{\varepsilon}_{ij} = B^e_{ijkl}\dot{\sigma}_{kl}+\frac{3}{2}\left(\frac{1}{E_Q}-\frac{1}{E}\right)I_{ijkl}\dot{\sigma}_{kl}+\left(\frac{3}{2\bar{\sigma}}\right)^2\left(\frac{1}{E_t}-\frac{1}{E_Q}\right)\dot{J}_2\sigma'_{ij}. \tag{69}$$

Assuming that

$$\frac{1}{2G_Q} = \frac{1+\mu_Q}{E_Q} = \frac{1+\mu}{E}+\frac{3}{2}\left(\frac{1}{E_Q}-\frac{1}{E}\right) \tag{70}$$

and

$$\frac{1}{h_Q} = \left(\frac{3}{2\bar{\sigma}}\right)^2\left(\frac{1}{E_t}-\frac{1}{E_Q}\right) \tag{71}$$

the inversion of eqn (69) is

$$\dot{\sigma}_{ij}=\frac{E_Q}{1+\mu_Q}\left\{\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\frac{\mu_Q}{1-2\mu_Q}\delta_{ij}\delta_{kl}-\frac{1}{q_Q}\sigma'_{ij}\sigma'_{kl}\right\}\dot{\varepsilon}_{kl} \tag{72}$$

where

$$q_Q=\frac{1+\mu_Q}{E_Q}+2J_2. \tag{73}$$

From the above comparison, it is clearly shown that all the theories, except J_2F theory, do not obey the orthogonality rule. It is also interesting to note that the inversion forms of these constitutive relations are analogous to that of the J_2F theory, but the only difference is that the elastic modulus and the Poisson ratios in these constitutive equations are functions of deformation (strain).

It should be mentioned that during the whole deformation, the classical J_2F theory of plasticity with a smooth yield surface always obeys the orthogonality rule. Whenever buckling prediction involves an abrupt change in the relative proportions of the components of the stress increments, the bifurcation load or deformation from any of the classical flow theories overestimates findings from buckling experiments, in some instances by a considerable amount (Christofferson and Hutchinson, 1979). On the other hand, the J_2D, J_2C and J_2G theories do not obey the orthogonality rule throughout the whole deformation process from initial plastic loading to fracture. The advantage of these theories is that in the vicinity of the criterial instability position of materials, or during the localized evolution after instability, these constitutive equations describe well the plastic behaviors and localization. However, the orthogonality plastic flow and the corresponding smooth yield surface are applicable to the large part of the plastic deformation before instability. Comparing these plasticity theories, the present QJ_2FC theory builds a reasonable link between these two kinds of theories, and a continuous transition from the orthogonality rule to the non-orthogonality during plastic deformation is realized. Furthermore, the parameters C_0 and $\bar{\varepsilon}_0^{p*}$ are of clear physical meanings on phenomenological point of view, and can be easily determined according to different material parameters, loading or deformation conditions and critical instability criterions.

2.4. *Extensions into the finite strain problems*

In finite elastoplasticity, the basic kinematic assumption is the multiplicative decomposition of total deformation gradient. The total strain description in the small strain range must be coincided with the finite strain formulation of elastic–plastic solids. Although some new descriptions of strain rate for the finite elastoplasticity have been proposed by Green and Naghdi (1965), Hill and Rice (1972), Duszek and Perzyna (1991, 1993), and Shieck and Stumpf (1995), in order to directly compare the simulated results of the present theory with that of Budiansky (1959), Christofferson and Hutchinson (1979) and Gotoh (1985), the well-known ZJ rate is still used in the present analysis. Considering that most of the ductile metal materials in plastic forming are of a character of infinite elasticity and finite plasticity.

A simple remedy is to replace the strain rate $\dot{\varepsilon}_{ij}$ by the logarithmic–strain rate, or deformation rate, v_{ij} defined relative to current material configuration. In this case, the assumption of eqn (1) relative to small strain range can be considered to be suited to finite strain case.

Then, the general description of infinite constitutive laws must introduce such stress measures that implies the effects of rigid displacement and keep so-called objective invariability relative to reference configuration. Considering the material as discussed above is elastically compressible, the most straightforward scheme is to use the above formulas unaltered with σ_{ij} taken to be the Kirchhoff stress tensor and $\dot{\sigma}_{ij}$ as its Jaumann rate. Cartesian components in the above equations may be converted to components referred to base vectors deforming with the materials. Consistent with the above proposal, E_S and E_t should be the secant and tangent moduli of the uniaxial stress–logarithmic–strain curve at the current stress state σ_{ij}. With this interpretation, the constitutive equations (14), (26) and (27) will coincide with the proposal made in J_2D, J_2C and J_2G theories.

3. NUMERICAL TESTS

Several typical examples including plane strain tension and uniaxial tension of anisotropic sheet metals are numerically calculated with the present QFC theory and other J_2-class theories, in order to show the application of the proposed theory and to compare it with the other theories.

3.1. *Plane strain tension*

In this section, the simulated materials are assumed to be isotropic plasticity. Four J_2 types of constitutive theories are applied to calculate the plastic deformation and the development of localization for the plane strain tension of a black with an initial surface imperfection, in order to compare the deformation behaviors of the loading and unloading among the different theories.

Figure 2(a) shows the geometry and loading condition of a plane strain block (a quarter) with x_3 infinitive, and Fig. 2(b) is its element meshing. $\Delta = f_0D$ represents the initial geometrical imperfection where f_0 is an imperfection parameter. The range of the initial imperfection is in $L/3$. The stress and strain relation of material is assumed to obey the hardening law: $\bar{\sigma} = K_0\bar{\varepsilon}^n$ where n is taken as 0.0625 and $E = 200$ GPa, $\mu = 0.33$. The other parameters are taken as: $f_0 = 0.005$, $C_0 = 0.9$, $a_0 = n/20$, $\theta_0 = 25°$ and $\delta_Q = 0.00025$.

Figure 3(a)–(d) show the deformed specimens and meshes at $\Delta L/L = 0.186$ calculated with different constitutive theories. Figure 4(a) and (b) are, respectively, the corresponding $\bar{\varepsilon}_{max}/\bar{\varepsilon}_{av} \sim \bar{\varepsilon}_{av}$ and the modulus evolutional curves of different theories, where $\bar{\varepsilon}_{max}$ is the maximum effective strain among all the elements and $\bar{\varepsilon}_{av}$ is the average one of the whole specimen. It is clearly shown that the above deformed configurations correspond to the localization stage. Numerically, the localization calculated with J_2F theory is much later than that with J_2D theory and the localizations calculated with QJ_2FC and J_2G theories are located between them. Also, a severe shear band is shown in Fig. 3(c) for J_2D theory and no shear band forms in Fig. 3(a) for J_2F theory, and the shear bands for QJ_2FC and J_2G theories are moderate. All three kinds of theories, except J_2F, observe continuous decreases of quasi-elastic moduli with strain. These variations of quasi-elastic moduli are related to the formations of shear band, and there is no shear band for J_2F theory which shows a constant elastic modulus during deformation. Finally, all the three quasi-elastic moduli come back to the Young's modulus when evident elastic unloading occurs.

In the above calculations, a small strain hardening exponent of $n = 0.0625$ has been chosen in order to show evidently the formation of shear band (even though, there is no

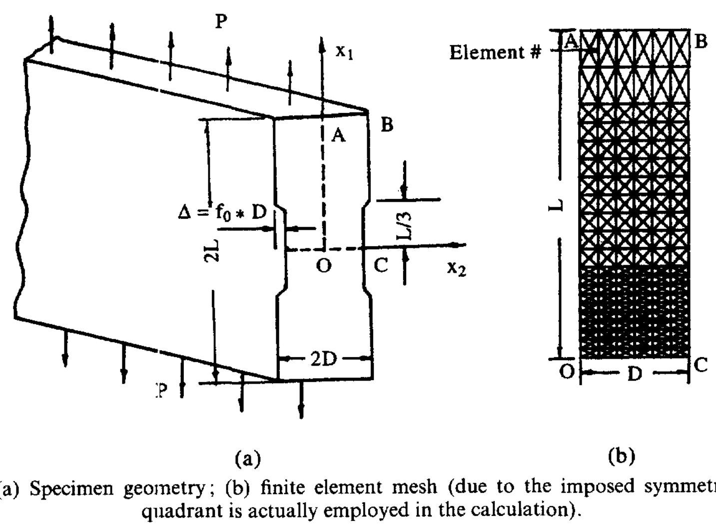


Fig. 2. (a) Specimen geometry; (b) finite element mesh (due to the imposed symmetry only a quadrant is actually employed in the calculation).

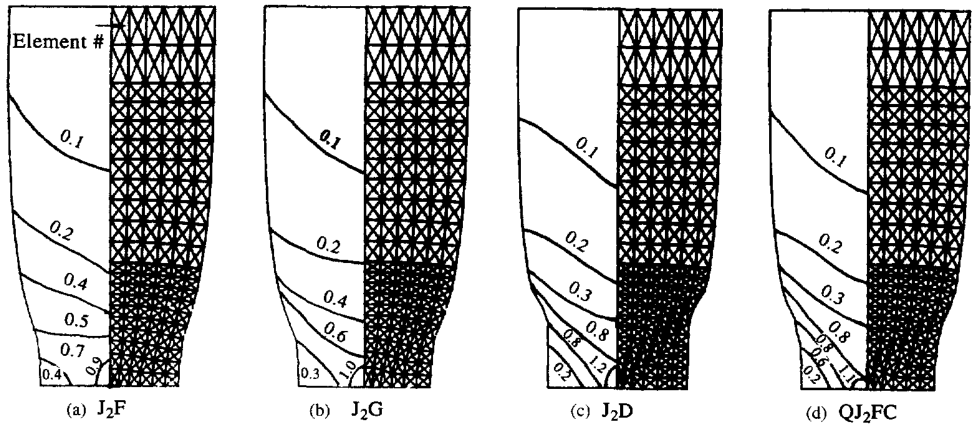


Fig. 3. The deformed mesh at $\Delta L/L = 0.186$ for four constitutive theories with $n = 0.0625$ and $f_0 = 0.005$.

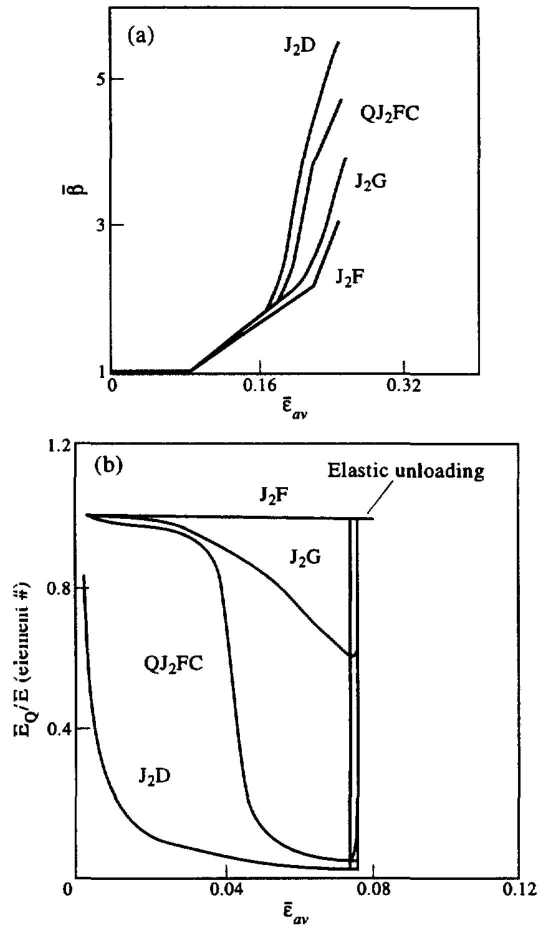


Fig. 4. (a) $\bar{\varepsilon}_{max}/\bar{\varepsilon}_{av} - \bar{\varepsilon}_{av}$ curve obtained by the four constitutive theories; (b) modulus evolutional curves with the average effective strain $\bar{\varepsilon}_{av}$ for the four constitutive theories during the whole deformation from plastic loading to elastic unloading.

shear band for J_2F). In fact, both n and f_0 have influences on the formation of shear band. The formation of shear band or not can be generally considered as a distinction between localized necking (or fracture) and diffuse necking. In order to study the influence of both parameters n and f_0 on the formation of shear band, two simple criteria are defined. One is the fracture criterion, that is, $\bar{\beta} = \bar{\varepsilon}_{max}/\bar{\varepsilon}_{av}$. From numerous calculations, it is assumed that

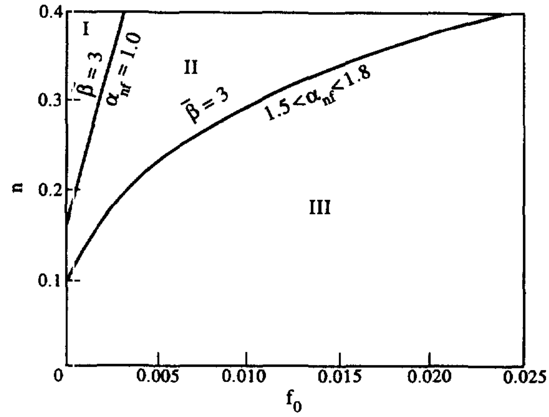
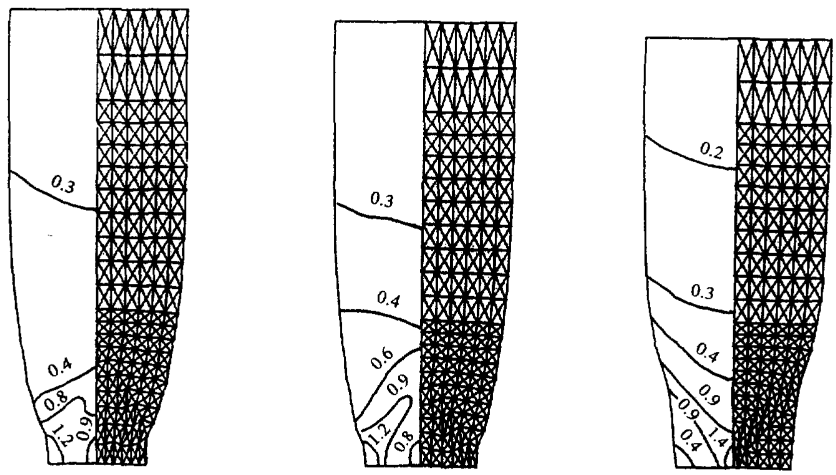


Fig. 5. Fracture mechanism diagram for plane-strain tension.



(a) $\Delta L/L = 0.388\ f_0 = 0.0001$ Diffuse necking (b) $\Delta L/L = 0.357\ f_0 = 0.001$ Mixture necking (c) $\Delta L/L = 0.32\ f_0 = 0.02$ Localized necking

Fig. 6. Three typical deformed configurations before fracture simulated with the QJ_2FC theory ($\bar{\beta} = 3.0$): (a) diffuse necking; (b) mixture necking; (c) localized necking.

fracture occurs when $\bar{\beta} \geqslant 3.0$. The other criterion is $\alpha_{nf} = \bar{\varepsilon}^A_{av}/\bar{\varepsilon}^B_{av}$ where $\bar{\varepsilon}^A_{av}$ is the average effective strain within shear band and $\bar{\varepsilon}^B_{av}$ is that out of shear band, both values are all measured within the necking region (i.e. the region in the initial imperfection region). It is understood that $\alpha_{nf} = 1.0$ means no shear band and a larger value of α_{nf} than 1.0 means more serious shear band. From a great number of calculations with the QJ_2FC theory and a careful induction, a fracture mechanism diagram for plane strain tension is shown in Fig. 5, which reflects the influences of both n and f_0 on the fracture mode. In this figure, there are three regions. Region III corresponds to $\bar{\beta} \geqslant 3.0$ and $\alpha_{nf} > 1.5$, and evident shear bands form in this region. Region I corresponds to $\bar{\beta} \geqslant 3.0$ and $\alpha_{nf} \simeq 1.0$, and there is no shear band. Region II is a mixture region where moderate or slight shear band forms. In fact, region I is the region of diffuse necking where specimen will finally fracture in the cross-section at the center; Region III is the region of localized necking where fracture occurs along shear band. Figure 6(a)–(c) show three typical deformed configurations representing the localized fracture, diffuse fracture and mixture fracture, respectively.

3.2. *Localized deformation of anisotropic sheet metal under uniaxial tension*

An example of localized deformation process of a sheet under uniaxial tension calculated with the *QFC* theory is given. The material parameters are chosen as: $n = 0.2$, $E = 200{,}000$ MPa and $\mu = 0.3$. Other material and geometrical parameters and element

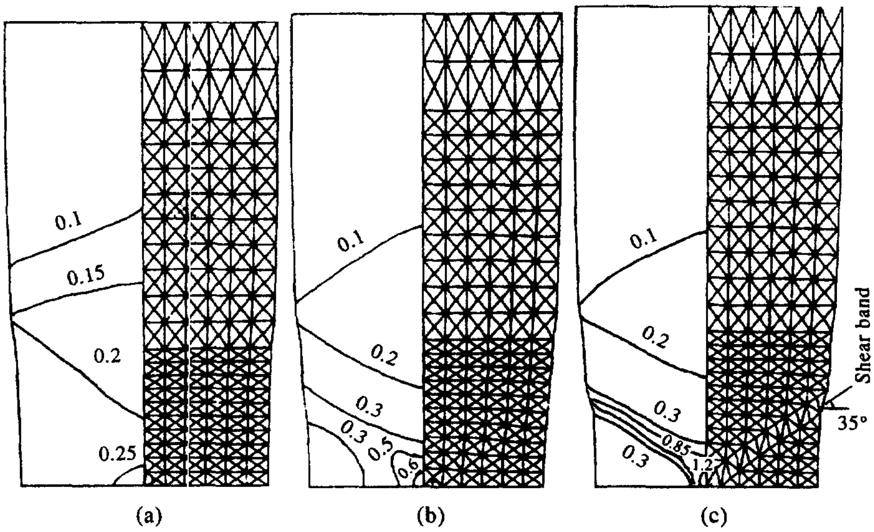


Fig. 7. Necking instability and localized deformation process under plane-stress tension: (a) $\Delta L/L = 0.128$; (b) $\Delta L/L = 0.152$; (c) $\Delta L/L = 0.175$.

meshing are the same with the Section 3.1. Figure 7(a)–(c) show the numerically calculated necking instability and localized deformation process under plane stress tension for different stages. When the stretching ratio $\Delta L/L$ reaches about 0.175, a clear shear band is formed, which declines about 35° with the abscissa. From Hill's localized instability theory, the shear instability direction or the zero-extension direction is expressed as (Lian and Baudelet, 1987):

$$\varphi^* = \tan^{-1}\left\{\left[\frac{H-\alpha(F+H)}{1-H\alpha}\right]^{1/2}\right\} \tag{74}$$

where

$$\alpha = \frac{\sigma_2}{\sigma_1}, \quad H = \frac{R_0}{1+R_0}, \quad F = \frac{R_0}{R_{90}(1+R_0)}. \tag{75}$$

R_0 and R_{90} are the ratios of transverse strain to thickness strain at rolling and transverse directions of sheet, respectively. For the case of uniaxial tension ($\sigma_2 = 0$) an isotropic material, $\varphi^* = 35.26$ is obtained from eqn (74). Therefore, it is seen that the shear band and its direction simulated with the present *QFC* theory is in agreement with the classical instability theory.

For anisotropic sheet, the Barlat and Lian (1989) yield function (a M-order yield function) is used:

$$f = |k_1+k_2|^{\mathrm{M}} + |k_1-k_2|^{\mathrm{M}} + \frac{c}{2-c}|2k_2|^{\mathrm{M}} = \frac{2}{2-c}\sigma_{\mathrm{e}}^{\mathrm{M}} \tag{76}$$

where

$$k_1 = \frac{\sigma_x+h_1\sigma_2}{2}, \quad k_2 = \left[\left(\frac{\sigma_x-h_1\sigma_2}{2}\right)^{1/2} + h_2^2\sigma_{xy}^2\right]^{1/2} \tag{77}$$

and

Table 1. Material parameters for three kinds of sheet metals

Materials	R_0	R_{45}	R_{90}	n	σ_0 (MPa)	E (MPa)
08Al	1.961	1.204	2.176	0.245	401	201,000
1Cr18Ni9Ti	0.935	1.544	1.067	0.396	404	202,000
D15	0.311	0.517	0.424	0.166	360	180,200

Table 2. The comparison of total elongation between experiments and simulation

	δ% (expt.)		δ% (predicted)	
Materials	0 deg.	90 deg.	0 deg.	90 deg.
08Al	42	42	37.7	38.5
1Cr18Ni9Ti	52	50	54.5	57.2
D15	17.6	19.5	14.5	17.2

$$\begin{cases} c = 2\left[\dfrac{R_0}{1+R_0}\dfrac{R_{90}}{1+R_{90}}\right]^{1/2}, \quad h_1 = \left[\dfrac{R_0}{1+R_0}\dfrac{1+R_{90}}{R_{90}}\right]^{1/2} \\ h_2 = \dfrac{\sigma_0}{\tau_0}\left[\dfrac{[(1+R_0)(1+R_{90})]^{1/2}}{2[(1+R_0)(1+R_{90})]^{1/2}+(2^{\mathrm{M}}-2)(R_0R_{90})^{1/2}}\right]^{1/\mathrm{M}} \end{cases} \tag{78}$$

with σ_0 and τ_0 being the uniaxial yield stress in the rolling direction and the shear yield stress, and the relation between strain and stress ratio has the following form

$$\rho = \frac{\dot{\varepsilon}_2}{\dot{\varepsilon}_1} = \frac{\left[\dfrac{|h_1\alpha|}{\alpha} - \dfrac{c}{2-c}\dfrac{|1-h_1\alpha|^{\mathrm{M}}}{1-h_1\alpha}\right]}{\left[1+\dfrac{c}{2-c}\dfrac{|1-h_1\alpha|^{\mathrm{M}}}{1-h_1\alpha}\right]}. \tag{79}$$

For the comparison between present theory with the experiments of uniaxial tension of sheet, three sheet materials have been used, i.e. 08Al steel (similar to AK steel), 1Cr18Ni9Ti stainless steel and D15 steel (Si–steel). Tensile test has been performed for these sheets. The material parameters are shown in Table 1. The total elongations at fracture for these three sheets are listed in Table 2. For theoretical simulation, the experimental values of parameters are used. The index M in the B-L yield function is taken 6 for 08Al and D15 steels (bcc metals) and 9 for 1Cr18Ni9Ti (fcc metal). The fracture criterion of $\bar{\beta} \geqslant 4.0$ is used here. The calculated total elongations with this fracture criterion for these three sheet materials are shown in Table 2.

It is seen that the predicted total elongations are generally in agreement with the experimental results for these materials. The shear band direction is also compared. For simplification, the case of plane isotropy is assumed, i.e. an average strain ratio of $\bar{R} = (R_0+2R_{45}+R_{90})/4$ is used. In this case, the B–L yield function reduces to the Hosford yield function. Figure 8 shows the variation of shear band direction with the average strain ratio calculated by the present theory. It is seen from Fig. 8 that the shear band direction increases rapidly with the increment of $\bar{R}$ before $\bar{R} \leqslant 0.5$ and increases very slowly after that. The experimental values of the three materials are also plotted on Fig. 8, which shows the same tendency.

4. CONCLUSIONS

The classical J_2F theory always obeys the normality law of plasticity theory, therefore, the vertex effect related to the plastic instability cannot be reflected by the J_2F theory.

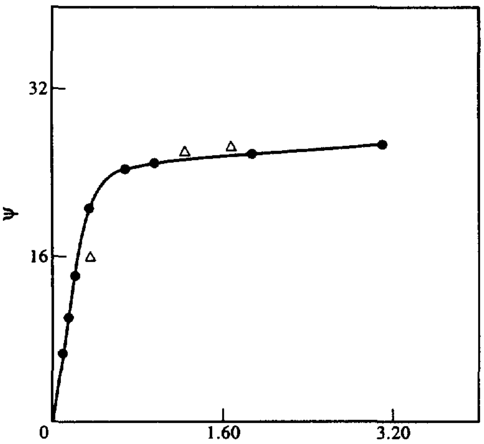


Fig. 8. Relation curve between the $\bar{R}$ value and shear band direction Ψ (△—experimental values, ●—calculated values).

Other classes of plasticity theories (for example J_2D, J_2C and J_2G theories) developed for describing the plastic instability or vertex effect do not obey the normality law during the whole deformation process. In order to modify the above both classes of plasticity theories, a relatively simple phenomenological quasi-flow corner theory for elastic plastic finite deformation has been proposed. By introducing a quasi-elastic modulus and a modulus variation function into the J_2F theory and the normality law, the present *QFC* theory obeys the normality law during homogeneous plastic deformation stage and realizes the gradual transitions from the normality law to the non-normality law and from plastic loading to elastic unloading. Therefore, both the classical J_2F theory and the J_2D type theories are included by the present theory.

The *QFC* theory is then applied to simulate the plastic deformation and localization processes of both plane strain tension and uniaxial tension of sheet. For plane strain case, both localized necking and diffuse necking can be simulated and well distinguished and a fracture mechanism diagram distinguishing the diffuse and localized neck according to the material parameter and initial geometry imperfection is presented. For the simulation of uniaxial tension of sheet metals, the predicted total elongation and the shear band direction with the *QFC* theory are in agreement with the experimental results of three sheet steels. The shear band direction simulated for isotropic case are also in coincidence with the result predicted with the classical instability theory.

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